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Relativistic scattering in a slowly varying external field

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Abstract. A low-frequency approximation for the scattering of a spinless charged particle by a spinless neutral target in the presence of an external electromagnetic field, originally derived for the case of a monochromatic plane wave of infinite extent, is generalised so that it applies to the more realistic case of a wave train of finite length. The case where the charged particle has spin $\frac{1}{2}$ and an anomalous magnetic moment is also treated. The field is taken to be slowly varying relative to the collision time, but the spectral composition of the field is otherwise arbitrary; the static limit, corresponding to a constant crossed field, is included as a special case. The assumption that the collision is essentially instantaneous is formulated in a gauge-invariant manner, and this provides the physical basis of the derivation. As in earlier versions of the low-frequency approximation, the approximate transition amplitude is expressed in terms of the on-shell amplitude for scattering in the absence of the field.

1. Introduction

In the course of an analysis of the connection between low-frequency approximations and the classical limit, Brown and Goble (1968) derived an approximation for the cross section for scattering of a charged scalar particle in the presence of a monochromatic external radiation field which is correct to all powers of the electric charge and which correctly provides the first two terms in the expansion of the exact cross section in powers of the frequency of the field. Non-relativistic versions of the Brown–Goble approximation have been developed subsequently within the context of potential scattering models (Kroll and Watson 1973, Mittleman 1980, Rosenberg 1981), and experimental investigations of electron–atom scattering in a laser field have been undertaken (Weingartshofer *et al* 1979). In view of the continuing interest in the problem of scattering in a laser field, we have thought it worthwhile to return to the relativistic model of the scattering adopted by Brown and Goble in order to gain further insight into the nature and domain of validity of the low-frequency approximation.

Closed-form solutions for the charged-particle motion in the field can be obtained for a plane-wave field of arbitrary spectral composition (Volkov 1935). As shown explicitly by Neville and Rohrlich (1971), wave packets can be constructed from linear superpositions of Volkov solutions. These packets serve as the appropriate asymptotic states in a description of the scattering process. The centre of the packet follows a classically determined trajectory and, for a wave train of finite length, spends only a finite amount of time in the region where the field is non-vanishing. These solutions have been adopted in the present work. Using gauge invariance as a guide, and requiring that the field be sufficiently slowly varying (a condition stated more precisely below), we have derived a generalised version of the low-frequency approximation.

Several earlier versions, both relativistic and non-relativistic, appropriate to scattering in a monochromatic field, are contained in the present result as special cases. One such case is the external-field version of the theorem derived by Low (1958) for spontaneous single-photon bremsstrahlung, obtained here by passing to the weak-field limit. The static limit, corresponding to a constant crossed field, is another special case of interest. It should be noted that the static limit is *not* obtained from the amplitude for scattering in a monochromatic field by allowing the frequency to approach zero. That limit is singular since the vector potential is unbounded in a zero-frequency field of infinite extent. In the present formalism, on the other hand, the wave train is of finite length and the static limit is well defined.

We first study the model adopted by Brown and Goble which involves the elastic scattering of two spinless particles, one charged and one neutral. We then treat the case where the charged particle has spin $\frac{1}{2}$ and anomalous moment μ_{A}. Analogous results could be obtained for other scattering systems. In particular, the electron–atom system, described by the non-relativistic Schrödinger equation, can be analysed along very similar lines with the aid of the gauge transformation technique described previously (Rosenberg 1981).

2. Formulation of the scattering problem

We consider here the elastic scattering of two spinless particles, one of charge e and mass m, the other neutral and of mass M. In the absence of the external field, and in the limit where the asymptotic solutions are plane waves $\exp(\mathrm{i}p\cdot x)$ of infinite extent $(p\cdot x \equiv \boldsymbol{p}\cdot\boldsymbol{x} - p^0x^0)$, the invariant amplitude for the transition changing the charged-particle momentum from p to p' while the target has its momentum changed from q to q' can be represented as

$$\mathcal{T}(p', q'; p, q) = \int \mathrm{d}^4x'\,\mathrm{d}^4x\,\exp(-\mathrm{i}p'\cdot x')\mathcal{T}_{q'q}(x', x)\exp(\mathrm{i}p\cdot x). \tag{2.1}$$

Here $\mathcal{T}_{q'q}(x', x)$ is defined as $\int \mathrm{d}^4y'\,\mathrm{d}^4y\,\exp(-\mathrm{i}q'\cdot y')\mathcal{T}(x', y'; x, y)\exp(\mathrm{i}q\cdot y)$, with $\mathcal{T}(x', y'; x, y)$ representing the collection of Feynman matrix elements in configuration space appropriate to the two-particle collision process. The relation

$$\mathcal{T}(p', q'; p, q) = (2\pi)^4\delta^4(P'-P)T(p', q'; p, q), \tag{2.2}$$

with $P = p+q$ and $P' = p'+q'$, expresses conservation of total momentum. The invariant T amplitude is assumed to have a non-singular off-mass-shell extension. The physical amplitude for the isolated two-particle system is obtained by imposing the on-shell conditions $p^2+m^2 = p'^2+m^2 = 0$. (The conditions $q^2+M^2 = q'^2+M^2 = 0$ are assumed to hold in all that follows.)

Suppose now that the scattering takes place in the presence of an external plane-wave field of arbitrary spectral composition and polarisation properties. The vector potential A_μ is taken to be a function of $u \equiv -n\cdot x$, with $n^2 = 0$ and $n^0 > 0$. One may think of such a field as representing a superposition of plane waves $\exp(\mathrm{i}k\cdot x)$ with $k = \omega n$ and ω arbitrary. The unique propagation direction is $\boldsymbol{n}$, and ω represents the angular frequency in a reference frame where $n^0 = 1$. (We use natural units, with $\hbar = \mathrm{c} = 1$). More generally, $A_\mu(u)$ is an arbitrary function of u subject to the condition

$$\partial_\mu A^\mu = -\mathrm{d}(n\cdot A)/\mathrm{d}u = 0.$$

This is satisfied by requiring that $n \cdot A = 0$. One still has the freedom of introducing a gauge transformation

$$A_\mu(u) \to A_\mu(u) + \partial_\mu \lambda(u) = A_\mu(u) - n_\mu \, \mathrm{d}\lambda/\mathrm{d}u,$$

which preserves the condition $n \cdot A = 0$. The potential is assumed here to vanish for $|u| > u_0$. Such a cut-off is unrealistically sharp, but is adopted to simplify the analysis.

With the field present, the plane wave describing the asymptotic motion of the charged particle is replaced by $\psi_p^{(+)}$ (or $\psi_{p'}^{(-)}$) for the initial (or final) state. These functions satisfy

$$[(-\mathrm{i}\,\partial - eA)^2 + m^2]\psi_p^{(\pm)}(x; A) = 0 \tag{2.3}$$

and can be expressed as

$$\psi_p^{(+)}(x; A) = \exp(\mathrm{i}p \cdot x) \exp\left(-\mathrm{i}\int_{-u_0}^{u} I_p(\bar{u})\,\mathrm{d}\bar{u}\right) \to \exp(\mathrm{i}p \cdot x), \qquad u < -u_0, \tag{2.4a}$$

$$\psi_{p'}^{(-)}(x; A) = \exp(\mathrm{i}p' \cdot x) \exp\left(\mathrm{i}\int_{u}^{u_0} I_{p'}(\bar{u})\,\mathrm{d}\bar{u}\right) \to \exp(\mathrm{i}p' \cdot x), \qquad u > u_0, \tag{2.4b}$$

where

$$I_p(u) = (2n \cdot p)^{-1}[2ep \cdot A(u) - e^2 A^2(u)]. \tag{2.5}$$

These Volkov plane waves should in fact be replaced by Volkov wave packets as discussed by Neville and Rohrlich (1971). Such a replacement will be understood in the following, rather than explicitly written out. The field-modified version of equation (2.1) is

$$\mathcal{T}(p', q'; p, q; A) = \int \mathrm{d}^4x' \, \mathrm{d}^4x \, \psi_{p'}^{(-)*}(x'; A) \mathcal{T}_{q'q}(x', x; A) \psi_p^{(+)}(x; A). \tag{2.6}$$

The construction of the modified kernel $\mathcal{T}_{q'q}(x', x; A)$ is discussed below.

3. Impact approximation

We now introduce an approximation for the kernel $\mathcal{T}_{q'q}(x', x; A)$ appearing in equation (2.6) based on the assumption that the characteristic time over which the field changes by an appreciable fraction of itself is large compared with the collision time. More precisely, let u_c be a measure of the collision time in the rest frame of the charged particle. We assume that

$$\frac{e^2}{m^2}\left(\frac{\mathrm{d}A}{\mathrm{d}u}\right)^2 u_c^2 \ll 1 \tag{3.1}$$

for all $|u| < u_0$. For the particular case of a monochromatic wave of frequency ω this condition is roughly equivalent to $(e^2/m^2)\ [A(u)]^2(\omega u_c)^2 \ll 1$, and is satisfied for $(e^2/m^2)A^2$ not much greater than unity. (In fact, inserting values of e and m appropriate to the electron, one finds that $(e^2/m^2)A^2$ barely reaches the order of magnitude unity for the strongest laser fields presently available.) In the crudest statement of the impact approximation for a non-resonant collision in a field satisfying the condition (3.1) one assumes that the collision takes place too rapidly for the field to have an appreciable effect on the intermediate states of the scattering process, so that

$\mathcal{T}_{q'q}(x', x; A) \cong \mathcal{T}_{q'q}(x', x)$. Since the field has ample time to act in initial and final states, the correct Volkov solutions must be used in equation (2.6). In the form just stated, the approximation fails to satisfy gauge invariance. Earlier work by Low (1958) and Brown and Goble (1968) suggests that an improved approximation can be obtained which restores gauge invariance and which allows one to construct the amplitude for scattering in the field from a knowledge of the on-shell field-free scattering amplitude. We will show below that the form

$$\mathcal{T}_{q'q}(x', x; A) \cong \exp[\mathrm{i}e\Lambda(x', x)]\mathcal{T}_{q'q}(x', x), \tag{3.2}$$

with

$$\Lambda(x', x) = (x'-x) \cdot \int_0^1 \mathrm{d}\eta\, A[u+\eta(u'-u)], \tag{3.3}$$

used in conjunction with equation (2.6), provides us with such an improved approximation. We will also verify that for the special case of a monochromatic wave the Brown–Goble low-frequency approximation is reproduced. At the same time we will confirm the validity of the Brown–Goble procedure based on an approximation rather similar to equation (3.2) but involving only the on-shell amplitude. That is, we introduce the field-free amplitude off the mass shell, as in equation (3.2), and specify conditions under which the off-shell components may be neglected.

To see how the approximate form (3.2) arises we examine the structure of the particle–field propagator $G(x, x'; A)$ which satisfies

$$\{[-\mathrm{i}\partial - eA(x)]^2 + m^2\}G(x, x'; A) = \delta(x-x'). \tag{3.4}$$

It has been shown (Schwinger 1951, Brown and Kibble 1964) that the propagator may be represented as

$$G(x', x; A) = \exp[\mathrm{i}e\Lambda(x', x)]\Delta_{\mathrm{c}}[x'-x; m^2 + \mathcal{M}^2(u', u)]. \tag{3.5}$$

$\Delta_{\mathrm{c}}(x'-x; m^2)$ is the free propagator for a spin-zero particle of mass m. The coordinate-dependent mass shift is given by

$$\mathcal{M}^2(u', u) = e^2 \int_0^1 \mathrm{d}\eta'\, A[u+\eta'(u'-u)]\left(A[u+\eta'(u'-u)] - \int_0^1 \mathrm{d}\eta\, A[u+\eta(u'-u)]\right). \tag{3.6}$$

In the form (3.5) $G(x', x; A)$ evidently satisfies the correct gauge transformation property (which follows directly from equation (3.4)) since $\mathcal{M}^2$ is gauge invariant and $\Lambda(x', x) \to \Lambda(x', x) - \lambda(u) + \lambda(u')$ for $A^\mu \to A^\mu - n^\mu\, \mathrm{d}\lambda(u)/\mathrm{d}u$. For our present purposes the essential property of $\mathcal{M}^2$, evident from its definition (3.6), is that it vanishes for $u' \to u$. To see how rapidly $\mathcal{M}^2$ vanishes in this limit, one expands A_μ in equation (3.6) about some fixed point in the range of integration. Ignoring second-derivative terms, one readily finds

$$\mathcal{M}^2(u', u) \cong \frac{e^2}{12}\left(\frac{\mathrm{d}A}{\mathrm{d}u}\right)^2 (u'-u)^2. \tag{3.7}$$

Physically, this behaviour reflects the fact that, over a time span short compared with a characteristic period of the field, the mass shift due to the interaction with the field cannot build up appreciably. Now we are studying the propagation *during* the collision.

Setting $(u'-u)^2 \sim u_c^2$ we have

$$\frac{\mathcal{M}^2}{m^2} \cong \frac{1}{12}\frac{e^2}{m^2}\left(\frac{dA}{du}\right)^2 u_c^2,$$

a quantity which we have assumed to be very small compared with unity. With $\mathcal{M}^2$ neglected we have the approximation

$$G(x', x; A) \cong \exp[ie\Lambda(x', x)]\Delta_c(x'-x; m^2) \tag{3.8}$$

for the propagator. To determine the effect of this approximation on the scattering amplitude we may visualise $\mathcal{T}(x', y'; x, y)$ as a collection of Feynman diagrams in configuration space. We assume that the two spinless bosons experience local interactions, with the propagation of the charged particle from one vertex to the next described by the form (3.8). As observed by Schwinger (1951), $\Lambda(x', x)$ can be redefined in such a way that it is represented by a path-independent integral, reducing to (3.3) when the path is taken to be a straight line. (See equations (3.17) and (4.28) of Schwinger's paper.) This path-independence property implies that as the entering charged-particle line passes continuously through the diagram it picks up the overall phase $e\Lambda(x', x)$, the *same* phase for each diagram, and that closed charged-particle loops have no additional phase factors associated with them. One then arrives at equation (3.2).

Insertion of the approximation (3.2) into equation (2.6) gives

$$\mathcal{T}(p', q'; p, q; A) \cong \int d^4x'\, d^4x\, \psi_{p'}^{(-)*}(x'; A) \exp[ie\Lambda(x', x)]\mathcal{T}_{q'q}(x', x)\psi_p^{(+)}(x; A). \tag{3.9}$$

The gauge invariance of this approximation is easily verified using the fact, evident from the definition (2.4*a*), that

$$\psi_p^{(+)}(x; A) \to \exp\{i[\lambda(u) - \lambda(u_0)]\}\psi_p^{(+)}(x; A)$$

for $A \to A - n(d\lambda/du)$. (Actually a constant phase is introduced in equation (3.9) by such a gauge transformation. This can be avoided by redefining the phase of $\psi_p^{(-)}(x; A)$, making this function identical to $\psi_p^{(+)}(x; A)$; it will be understood in the following that this has been done.)

As a convenient procedure for introducing the momentum-space representation of the field-free $\mathcal{T}$ amplitude appearing in equation (3.9), we make use of the identity

$$\begin{aligned}\psi_{p'}^{(-)*}(x'; A) &\exp[ie\Lambda(x', x)]\psi_p^{(+)}(x; A) \\ &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\int_{-\infty}^{\infty} d\bar{u}' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\int_{-\infty}^{\infty} d\bar{u}\, \exp\{i[\omega\bar{u} - S_p(\bar{u}) + Q_p(\omega, \bar{u}', \bar{u})\cdot x \\ &\quad -\omega'\bar{u}' + S_{p'}(\bar{u}') - Q_{p'}(\omega', \bar{u}', \bar{u})\cdot x']\},\end{aligned} \tag{3.10}$$

with

$$S_p(\bar{u}) = \int_{-u_0}^{\bar{u}} I_p(u'')du'' \tag{3.11}$$

and

$$Q_p(\omega, \bar{u}', \bar{u}) = p + \omega n - e\int_0^1 d\eta\, A[\bar{u} + \eta(\bar{u}' - \bar{u})]. \tag{3.12}$$

To verify this identity one simply performs the integrals over ω and ω' using the relation

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp[i\omega(\bar{u}+n\cdot x)] = \delta(\bar{u}+n\cdot x). \tag{3.13}$$

Recalling equations (2.1) and (2.2) we see that equation (3.9) may be rewritten in the form

$$\begin{aligned}\mathcal{T}(p', q'; p, q; A) &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} d\bar{u}' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d\bar{u}\, (2\pi)^4 \delta^4[P'-P+(\omega'-\omega)n] \\ &\times \exp\{i[\omega\bar{u} - S_p(\bar{u}) - \omega'\bar{u}' + S_{p'}(\bar{u}')]\} T[Q_{p'}(\omega', \bar{u}', \bar{u}), q'; Q_p(\omega, \bar{u}', \bar{u}), q].\end{aligned} \tag{3.14}$$

At this point we make use of the so-called light-like coordinates; they are defined in terms of the basis vectors n, the real orthonormal polarisation vectors ε_1 and ε_2, each orthogonal to n, and a fourth four-vector $\hat{n}$ which satisfies $\hat{n}^2 = 0$, $\hat{n}\cdot n = -1$ and $\hat{n}\cdot\varepsilon_i = 0$. An arbitrary vector V may be expanded as $V = V_1\varepsilon_1 + V_2\varepsilon_2 + V_v\hat{n} + V_u n$. The components in this basis are then $V_i = \varepsilon_i\cdot V$, $V_v = -n\cdot V$ and $V_u = -\hat{n}\cdot V$. In writing u for x_v we follow the notation of Neville and Rohrlich (1971) and Mitter (1975). Writing the δ function in equation (3.14) as

$$\delta(P_1'-P_1)\delta(P_2'-P_2)\delta(P_v'-P_v)\delta(\omega'-\omega+P_u'-P_u)$$

we may perform the integration over ω' and set

$$\omega' = \omega - (P_u' - P_u) \tag{3.15}$$

elsewhere in the integrand. Note that T in equation (3.14) is off the mass shell since the scalar variables

$$\xi = Q_p^2(\omega, \bar{u}', \bar{u}) + m^2, \tag{3.16a}$$

$$\xi' = Q_{p'}^2(\omega', \bar{u}', \bar{u}) + m^2 \tag{3.16b}$$

are non-vanishing in general.

4. Further analysis of the impact approximation amplitude

The vector potential, restricted only by the condition $n\cdot A = 0$, may be expanded as $A(u) = \varepsilon_1 A_1(u) + \varepsilon_2 A_2(u) + nA_u(u)$. We wish to take into account the slowly varying nature of the field, as characterised by the condition (3.1), to simplify the expression (3.14). Now the inequality (3.1) places a restriction on A_1 and A_2, but not on the component A_u. Since the expression (3.14) is gauge invariant we may, in the following, set $A_u = 0$ (radiation gauge); the gauge invariance of the final version will serve as a consistency check on the subsequent approximation procedure.

In accordance with the above remarks we ignore the variation of A between $\bar{u}$ and $\bar{u}'$ in equation (3.12) and write

$$Q_p(\omega, \bar{u}', \bar{u}) \cong Q_p(\omega, \bar{u}) \equiv p + \omega n - eA(\bar{u}), \tag{4.1a}$$

$$Q_{p'}(\omega', \bar{u}', \bar{u}) \cong Q_{p'}(\omega', \bar{u}') \equiv p' + \omega' n - eA(\bar{u}'). \tag{4.1b}$$

The scalar variables defined in equations (3.16) become, in this approximation,

$$\xi \cong 2n\cdot p[\omega - I_p(\bar{u})], \tag{4.2a}$$

$$\xi' \cong 2n\cdot p'[\omega' - I_{p'}(\bar{u}')], \tag{4.2b}$$

with ω' given by equation (3.15).

To make further progress we introduce more specific assumptions concerning the sizes of the relevant parameters in the problem. Let us consider two different regimes, one in which the function $S_p(u)$ is of order unity and the other in which it is large compared with unity. These have been referred to as the intermediate- and strong-coupling regimes, respectively, in an earlier analysis of electron–atom scattering in a monochromatic laser field (Rosenberg 1981). Now for $S_p \sim 1$, the case we consider first, its derivative $\mathrm{d}S_p/\mathrm{d}u = I_p(u)$ (which is a measure of the particle–field interaction energy) may, as a consequence of the assumed slow variation of the potential $A(u)$, be treated as a first-order quantity. Accordingly, we expect that the deviation off the mass shell is effectively small and can be accounted for by a Taylor series expansion of the T amplitude about zero values of the off-shell variables ξ and ξ' given in equation (4.2). Now ω in equation (4.2a) is an integration variable, ranging between infinite limits, but it may nevertheless be treated as a first-order quantity since the most significant region of integration corresponds to $\omega \sim I_p$. This statement is based on the observation that insertion of the factor $\omega - I_p(\bar{u})$ into the integrand in equation (3.14) gives rise to an integral which vanishes to first order, indicating that $\omega - I_p$ is effectively of second order. To check this we perform an integration by parts, with

$$\omega\ \exp(\mathrm{i}\omega\bar{u})\exp[-\mathrm{i}S_p(\bar{u})] \to [\mathrm{d}S_p(\bar{u})/\mathrm{d}\bar{u}]\exp(\mathrm{i}\omega\bar{u})\ \exp[-\mathrm{i}S_p(\bar{u})]$$

in the integrand. Surface terms introduce exponential factors which vary infinitely rapidly as subsequent integrations are performed and are neglected. We have also neglected a term involving $\mathrm{d}T/\mathrm{d}\bar{u}$ since $A(\bar{u})$ has only a first-order effect on the arguments of T, and differentiation generates only second-order corrections. (We assume here that T is a smooth function; the discussion would require modifications if scattering resonances were present.) The argument just given shows that $\omega - I_p$ is of second order 'on the average'; at the same time it shows that the first-order correction term in a series expansion of T about $\xi = 0$ vanishes. Second-order corrections will be ignored.

At this stage the T amplitude in equation (3.14) may be expressed as a function of the scalar variables $\nu = -Q_p(\omega, \bar{u})\cdot q$ and $\tau = -(q' - q)^2$ in addition to the off-shell variables ξ and ξ' introduced previously. We have shown that ξ may be set equal to zero to first order. Precisely the same reasoning which led us to replace ω by $I_p(\bar{u})$ in the ξ variable leads us to make the same replacement in the two other scalar variables in which ω appears. Thus ν becomes $-p(\bar{u})\cdot q$, where

$$p(\bar{u}) \equiv p - eA(\bar{u}) + nI_p(\bar{u}) \tag{4.3}$$

is the *classically* determined momentum for the charged particle in the field (Brown and Kibble 1964). Similarly, ξ' becomes, according to equations (4.2b) and (3.15),

$$\xi' = 2n\cdot p'[I_p(\bar{u}) - I_{p'}(\bar{u}') - (P'_u - P_u)]. \tag{4.4}$$

Since ω now appears only in the factor $\exp[\mathrm{i}\omega(\bar{u} - \bar{u}')]$, the integration over ω can be performed thereby introducing a δ function of $(\bar{u} - \bar{u}')$. The integration over $\bar{u}'$ is then trivial and we are left with

$$\mathcal{T} \cong (2\pi)^3 \delta(P_1' - P_1)\delta(P_2' - P_2)\delta(P_v' - P_v) \int_{-\infty}^{\infty} \mathrm{d}\bar{u} \, \exp\{\mathrm{i}[(P_u' - P_u)\bar{u} - S_p + S_{p'}]\} T. \tag{4.5}$$

Here T is a function of the scalar variables

$$\{\nu, \tau, \xi, \xi'\} = \{-p(\bar{u})\cdot q, -(q'-q)^2, 0, \xi'\},$$

with ξ' given by equation (4.4) and with $\bar{u}' = \bar{u}$. In the final step of the argument we expand T about $\xi' = 0$. The first-order correction term, involving $\xi'(\partial T/\partial \xi')$, vanishes. This follows from an integration by parts with

$$(P_u' - P_u) \exp\{\mathrm{i}[P_u' - P_u)\bar{u} - S_p + S_{p'}]\} \to (I_p - I_{p'}) \exp\{\mathrm{i}[(P_u' - P_u)\bar{u} - S_p + S_{p'}]\}.$$

Since we ignore second-order corrections, we may replace ξ' in equation (4.5) by zero. We then have the approximation

$$\begin{aligned}\mathcal{T}(p', q'; p, q; A) \\ &\cong (2\pi)^3 \delta(P_1' - P_1)\delta(P_2' - P_2)\delta(P_v' - P_v) \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}\bar{u} \, \exp\{\mathrm{i}[(P_u' - P_u)\bar{u} - S_p(\bar{u}) + S_{p'}(\bar{u})]\} T[p'(\bar{u}), q'; p(\bar{u}), q],\end{aligned} \tag{4.6}$$

with T now on the mass shell since $p^2(\bar{u}) + m^2 = p'^2(\bar{u}) + m^2 = 0$. Note that P_u is not conserved since the system is not invariant with respect to translations of the conjugate variable $u = -n \cdot x$. We have been working in the radiation gauge, but a transformation of the form $A_\mu(u) \to A_\mu(u) - n_\mu(\mathrm{d}\lambda/\mathrm{d}u)$ clearly leaves equation (4.6) invariant so that this gauge restriction may now be dropped.

In the foregoing discussion we have assumed that $S_p(\bar{u}) \sim 1$. We now consider the strong-coupling regime characterised by the condition $S_p(\bar{u}) \gg 1$ for $|\bar{u}| < u_0$. Since $S_p(\bar{u})$ vanishes for $|\bar{u}| > u_0$, we treat the two regions of integration separately. In the region $|\bar{u}| > u_0$ the previous argument, based as it was on the weakness of the interaction strength, can be taken over unchanged. For $|\bar{u}| < u_0$, however, the series expansion technique used earlier is inapplicable. Instead we note that with $S_p(\bar{u}) \gg 1$ the exponential will be varying rapidly as a function of $\bar{u}$, and the dominant contribution to the integral will come from the region where the phase is stationary. The stationary phase condition is $\omega = I_p(\bar{u})$, which is just the condition arrived at, by different reasoning, in the earlier treatment of the intermediate-coupling case. The previous argument can then be applied, *mutatis mutandis*, with the result that equation (4.6) is established for both the intermediate- and strong-coupling regimes.

5. Some special cases

(i) To make contact with previous work let us apply the approximation (4.6) to the case of a monochromatic plane wave of frequency ω. In addition to the basic low-frequency condition $\omega u_c \ll 1$ we assume that $\omega u_0 \gg 1$. According to this latter condition the field goes through many oscillations during the transit of the laser pulse through the scattering region; the cut-off then plays no essential role and for simplicity it will be removed. (To prevent the appearance of an infinite phase factor in equation (3.11) we redefine the integral by dropping the contribution from the lower limit and then letting $u_0 \to \infty$.) The vector potential is taken to be

$$A_\mu(\phi) = \mathscr{A}_\mu \exp(\mathrm{i}\phi) + \mathscr{A}_\mu^* \exp(-\mathrm{i}\phi), \tag{5.1}$$

where $\mathscr{A}_\mu$ is a constant amplitude and we have expressed A as a function of the phase $\phi = -\omega u$ rather than of u. In terms of the propagation vector $k = \omega n$ we have $\phi = k \cdot x$. In place of equation (3.11) we now write

$$S_p(\phi) = f_p(\phi) + \frac{e^2}{k \cdot p} \mathscr{A} \cdot \mathscr{A}^* \phi, \tag{5.2a}$$

with

$$f_p(\phi) = \mathrm{i}e\left(\frac{p \cdot \mathscr{A}}{k \cdot p} \exp(\mathrm{i}\phi) - \frac{p \cdot \mathscr{A}^*}{k \cdot p} \exp(-\mathrm{i}\phi)\right) - \frac{\mathrm{i}e^2}{4k \cdot p}[\mathscr{A}^2 \exp(2\mathrm{i}\phi) - \mathscr{A}^{*2} \exp(-2\mathrm{i}\phi)]. \tag{5.2b}$$

Let us now introduce the Fourier expansion

$$\exp\{\mathrm{i}[f_{p'}(\phi) - f_p(\phi)]\} T[p'(\phi), q'; p(\phi), q] = \sum_{r=-\infty}^{\infty} \exp(\mathrm{i}r\phi) T^{(r)}, \tag{5.3}$$

with inverse

$$T^{(r)} = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \exp(-\mathrm{i}r\phi) \exp\{\mathrm{i}[f_{p'}(\phi) - f_p(\phi)]\} T[p'(\phi), q'; p(\phi), q] \tag{5.4}$$

and with

$$p(\phi) = p - eA(\phi) + (2n \cdot p)^{-1}[2ep \cdot A(\phi) - e^2 A^2(\phi)]. \tag{5.5}$$

The integration over $\bar{u}$ in equation (4.6) may now be performed using the representation (5.3) with $\phi \to -\omega\bar{u}$ on the right-hand side. The result is

$$\mathscr{T}(p', q'; p, q; A) \cong \sum_{r=-\infty}^{\infty} (2\pi)^4 \delta^4(\tilde{p}' + q' - \tilde{p} - q - rk) T^{(r)}, \tag{5.6}$$

where

$$\tilde{p}_\mu = p_\mu - \frac{e^2 \mathscr{A} \cdot \mathscr{A}^*}{k \cdot p} k_\mu. \tag{5.7}$$

$T^{(r)}$ may be interpreted as the amplitude for scattering with the absorption of r photons if r is positive, or with the stimulated emission of $-r$ photons if r is negative; equation (5.4) represents a low-frequency approximation for this amplitude. This formula does not appear explicitly in the original treatment of this problem by Brown and Goble, who were primarily interested in deriving an expression for the total cross section in order to establish a classical correspondence. However, as shown subsequently (Rosenberg 1980), equation (5.4) can easily be derived from the underlying soft-photon approximation introduced by those authors. We note that a non-relativistic analogue of equation (5.4) has been obtained (Rosenberg 1981) for electron–atom scattering in a laser field.

(ii) Since we have placed a lower limit $\omega \gg u_0^{-1}$ on the frequency in the above derivation, the result cannot be used to study the limiting case of scattering in a constant crossed field. Indeed, if the electric field is to be non-vanishing in the limit $\omega \to 0$, the amplitude $\mathscr{A}_\mu$ in equation (5.1) will be unbounded in that limit. There is no difficulty, however, in using the general result (4.6) to treat this case. The appropriate choice of vector potential in the radiation gauge is, for $|u| < u_0$, $\boldsymbol{A} = -\boldsymbol{E}u$, with $\boldsymbol{E}$ a constant vector orthogonal to the unit vector $\boldsymbol{n}$. The electric and magnetic field vectors are $\boldsymbol{E}$ and $\boldsymbol{n} \times \boldsymbol{E}$, respectively. The presence of the cut-off prevents the vector potential from growing without bound.

(iii) In the strong-coupling regime the dominant contribution to the integral comes from a region near the point of stationary phase, determined by the condition

$$I_{p'}(\bar{u}) - I_p(\bar{u}) = -(P'_u - P_u). \tag{5.8}$$

Here we assume that equation (5.8) can in fact be satisfied for some $\bar{u}$, call it u_{sp}, in the region between $-u_0$ and u_0. Equation (4.6) may be simplified by evaluating the momenta $p(\bar{u})$ and $p'(\bar{u})$ at $\bar{u} = u_{\mathrm{sp}}$ and removing the T amplitude from underneath the integral sign. The value of $p(u_{\mathrm{sp}})$ can be determined rather easily in the special case of a linearly polarised field, with $A_0 = 0$ and $\boldsymbol{A}(u) = \boldsymbol{\varepsilon} a(u)$; we also go to the non-relativistic limit where $|\boldsymbol{p}|$ and $|\boldsymbol{p}'|$ are small compared with m. We then have

$$p(u_{\mathrm{sp}}) = p - eA(u_{\mathrm{sp}}) + nI_p(u_{\mathrm{sp}}) \cong p - eA(u_{\mathrm{sp}}), \tag{5.9}$$

ignoring a correction term of order $|\boldsymbol{p}|/m$. In this same approximation we have

$$I_{p'}(u_{\mathrm{sp}}) - I_p(u_{\mathrm{sp}}) \cong m^{-1} e(\boldsymbol{p}' - \boldsymbol{p}) \cdot \boldsymbol{\varepsilon} a(u_{\mathrm{sp}}). \tag{5.10}$$

The stationary phase condition then gives

$$a(u_{\mathrm{sp}}) = -\frac{m(P'_u - P_u)}{e(\boldsymbol{p}' - \boldsymbol{p}) \cdot \boldsymbol{\varepsilon}}. \tag{5.11}$$

This result allows us to determine $\boldsymbol{p}(u_{\mathrm{sp}}) = \boldsymbol{p} - e\boldsymbol{\varepsilon} a(u_{\mathrm{sp}})$ and $\boldsymbol{p}'(u_{\mathrm{sp}}) = \boldsymbol{p}' - e\boldsymbol{\varepsilon} a(u_{\mathrm{sp}})$, with $p_0(u_{\mathrm{sp}}) \cong m$, $p'_0(u_{\mathrm{sp}}) \cong m$. Accordingly, the approximation (4.6) becomes

$$\begin{aligned} &\mathcal{T}(p', q'; p, q; A) \\ &\quad \cong (2\pi)^3 \delta(P'_1 - P_1)\delta(P'_2 - P_2)\delta(P'_v - P_v) T[p'(u_{\mathrm{sp}}), q'; p(u_{\mathrm{sp}}), q] \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}\bar{u} \exp\left[\mathrm{i}\left((P'_u - P_u)\bar{u} + \int_{-u_0}^{\bar{u}} \frac{e}{m} (\boldsymbol{p}' - \boldsymbol{p}) \cdot \boldsymbol{\varepsilon} a(u') \mathrm{d}u'\right)\right]. \end{aligned} \tag{5.12}$$

Further specialisation to the case of a monochromatic wave leads to the version of the low-frequency approximation derived by Kroll and Watson (1973) in a potential scattering model.

6. Inclusion of spin effects

The methods developed above are now applied to the problem of the scattering of a particle of charge e, spin $\frac{1}{2}$, and anomalous magnetic moment μ_{A} from a neutral scalar particle in the presence of the field. The only new feature which needs emphasis is the effect of spin on the asymptotic states. In the absence of the external field these states satisfy the Dirac equation

$$(\gamma \cdot \partial + m) \exp(\mathrm{i}p \cdot x)\psi(p) = 0, \tag{6.1a}$$

$$(\mathrm{i}\gamma \cdot p + m)\psi(p) = 0, \qquad p^2 + m^2 = 0. \tag{6.1b}$$

In place of equation (2.2) we now have

$$\mathcal{T}(p', q'; p, q) = (2\pi)^4 \delta(P' - P)\bar{\psi}(p')T(p', q'; p, q)\psi(p). \tag{6.2}$$

As we have seen, the off-shell extension of the T matrix is relevant in the external-field problem; it turns out to be useful to introduce the projection operators

$$R_{\pm}(p) = \frac{\pm \mathrm{i}\gamma \cdot p + W}{2W}, \qquad W = (-p^2)^{1/2}, \tag{6.3}$$

with $R_+ + R_- = 1$. We may write, quite generally,

$$T = R_-(p')T_{--}R_-(p) + R_-(p')T_{-+}R_+(p) + R_+(p')T_{+-}R_-(p) + R_+(p')T_{++}R_+(p). \tag{6.4}$$

Evidently, it is only the component T_{--} which contributes to the amplitude (6.2) for $p^2 + m^2 = p'^2 + m^2 = 0$.

In the presence of the field the asymptotic states are chosen as solutions of

$$[\gamma \cdot (\partial - \mathrm{i}eA) + m - \tfrac{1}{2}\mu_{\mathrm{A}}\sigma_{\mu\nu}F^{\mu\nu}]\psi_p(x; A) = 0, \tag{6.5}$$

where $F^{\mu\nu}$ is the field tensor and $\sigma_{\mu\nu} = \mathrm{i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2$. Here we ignore the distinction between $\psi_p^{(+)}$ and $\psi_p^{(-)}$, to simplify notations. The solution to equation (6.5) has the form

$$\psi_p(x; A) = \exp(\mathrm{i}p \cdot x) \exp[-\mathrm{i}S_p(u)]\chi_p(u), \tag{6.6}$$

with

$$\chi_p(u) = J_p(u)\left(1 + \frac{e}{2n \cdot p}\gamma \cdot n\gamma \cdot A\right)\psi(p). \tag{6.7}$$

The function J_p can be constructed using the method described by Becker and Mitter (1974) and by Becker (1975). Explicit solutions are available for the case of linear polarisation, and for circular polarisation in the limit of a monochromatic wave train of infinite extent. More generally, one must solve, in some approximation, a set of coupled first-order differential equations (see equation (22) in the paper by Becker and Mitter) to obtain this function. In the present context the important property of the exact solution to be noted is the relation

$$[\mathrm{i}\gamma \cdot p(u) + m]\chi_p(u) = 0, \tag{6.8}$$

where $p(u)$ is the 'classical' momentum defined in equation (4.3). As will be seen, this relation guarantees that only the physical component T_{--} of the T matrix appears in the final version of the low-frequency approximation. Equation (6.8) may be verified directly, for example, by making use of the functional form of χ_p as expressed in equation (3.4) of the paper by Becker (1975).

For $\mu_{\mathrm{A}} = 0$ we have $J_p = 1$ and equation (6.6) reduces to the usual Volkov solution for a Dirac particle. In many cases of interest an approximation for J_p correct to first order in μ_{A} will be adequate. The coupled differential equations defining J_p are easily solved in this approximation. With the integration constants chosen to guarantee that $J_p \to 1$ for $\mu_A \to 0$ and that J_p contains no explicit dependence on the cut-off value u_0 (and hence is appropriate to the case of an infinitely long wave train), we find

$$J_p \cong 1 - \mu_{\mathrm{A}}\left[-\mathrm{i}\gamma \cdot \left(A - n\frac{(p - eA) \cdot A}{n \cdot p}\right) + \frac{m}{n \cdot p}\gamma \cdot n\gamma \cdot A\right]. \tag{6.9}$$

Before continuing any further let us review the basis for the approximation procedure to be used:

(i) The dominant effect of the field lies in the appearance of the phase factor S_p in the asymptotic states. This is most easily seen by specialising to the case of a field of well defined frequency ω. The integration in equation (3.11) introduces a factor ω^{-1}, a manifestation of the well known infrared near-singularity.

(ii) We also account for contributions of the type

$$L^{(0)}=\int_0^1 \mathrm{d}\eta\, A[u+\eta(u'-u)], \tag{6.10}$$

which appears, for example, in equation (3.3). For $|u'-u|\leqslant u_c$, a restriction appropriate to a discussion of the intermediate-state propagator, the integrand in equation (6.10) is very nearly constant over the domain of integration, so that no near-singularity can develop; the integral would be of order ω^0 for a monochromatic field.

(iii) We neglect terms of the type

$$L^{(1)}=A(u)-\int_0^1 \mathrm{d}\eta\, A[u+\eta(u'-u)] \tag{6.11}$$

for $|u'-u|\leqslant u_c$. Such a term appears, for example, in equation (3.6); it is of order $\mathrm{d}A/\mathrm{d}u$, or, in the monochromatic case, of order ω^1.

In order to arrive at an approximation analogous to that derived above for the spin-zero case, we must examine the spinor propagator in the presence of the field. The representation of this propagator derived by Becker (see his equation (5.5)) is convenient for this purpose. The propagator, which satisfies

$$[\gamma\cdot(\partial-\mathrm{i}eA)+m-\tfrac{1}{2}\mu_A\sigma_{\mu\nu}F^{\mu\nu}]G(x,x';A)=\delta(x-x'), \tag{6.12}$$

has a fairly complicated structure. However, in the approximation that terms of the same order as that shown in equation (6.11) (i.e. terms of order $\mathrm{d}A/\mathrm{d}u$) are neglected, we find that

$$G(x',x;A)\cong \exp[\mathrm{i}e\Lambda(x',x)]S_c(x'-x;m), \tag{6.13}$$

where S_c is the free spin-$\frac{1}{2}$ propagator. We may then follow the argument given previously, which led to the approximation (4.6), to obtain a similar form, in which the T amplitude in equation (4.6) is replaced by $\bar{\chi}_{p'}(\bar{u})T_{--}[p'(\bar{u}),q';p(\bar{u}),q]\chi_p(\bar{u})$. Let us emphasise that by virtue of equation (6.8) and its adjoint only the physical component T_{--} in the expansion (6.4) enters into our final result.

As a check on the formalism we may examine the weak-field limit and compare the approximate transition amplitude with that obtained for single-photon spontaneous bremsstrahlung by Low (1958). With terms of order A^2 neglected, equation (6.9) becomes, after a slight rearrangement,

$$J_p\cong 1-\frac{\mu_A}{2n\cdot p}[(-\mathrm{i}\gamma\cdot p+m)\gamma\cdot n\gamma\cdot A-\gamma\cdot n\gamma\cdot A(-\mathrm{i}\gamma\cdot p-m)]. \tag{6.14}$$

Equation (6.7), in this linear approximation, is then

$$\chi_p\cong\left(1+\frac{\mu_A}{2n\cdot p}(\mathrm{i}\gamma\cdot p-m)\gamma\cdot n\gamma\cdot A+\frac{e}{2n\cdot p}\gamma\cdot n\gamma\cdot A\right)\psi(p). \tag{6.15}$$

Assuming a monochromatic field of frequency ω we may determine the approximate amplitude for scattering with the emission of a single laser photon by setting $r=-1$ in the spinor analogue of equation (5.4). This amplitude is readily determined to be $\mathcal{A}^*_\mu M^\mu$, where M^μ appears as equation (3.16) in Low's paper.

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References

Becker W 1975 *J. Phys. A: Math. Gen.* **8** 160–70
Becker W and Mitter H 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1266–73
Brown L S and Goble R L 1968 *Phys. Rev.* **173** 1505–16
Brown L S and Kibble T W B 1964 *Phys. Rev.* **133** A705–19
Kroll N M and Watson K M 1973 *Phys. Rev.* A **8** 804–8
Low F 1958 *Phys. Rev.* **110** 974–7
Mitter H 1975 *Acta Phys. Austriaca, Suppl.* **14** 397–468
Mittleman M H 1980 *Phys. Rev.* A **21** 79–84
Neville R A and Rohrlich F 1971 *Phys. Rev.* D **3** 1692–707
Rosenberg L 1980 *Phys. Rev.* D **22** 527–9
——1981 *Phys. Rev.* A **23** 2283–92
Schwinger J 1951 *Phys. Rev.* **82** 664–79
Volkov D M 1935 *Z. Phys.* **94** 250–60
Weingartshofer A, Clarke E M, Holmes J K and Jung C 1979 *Phys. Rev.* A **19** 2371–6